Non-Uniform Cellular Automata *

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Abstract. In this paper we begin the study the dynamical behavior of non-uniform cellular automata and compare it to the behavior of "classical" cellular automata. In particular we focus on surjectivity and equicontinuity.

1 Introduction and Motivations

Cellular automata (CA) are a well-known formal model for complex systems that is used in many scientific fields [1–3]. Uniformity is one of the main characteristics of this model. Indeed, a cellular automaton is made of identical finite automata arranged on a regular lattice. The state of each automaton is updated by a local rule on the basis of the state of the automaton itself and of the one of a fixed set of neighbors. At each time-step, the same (here comes uniformity) local rule is applied to all finite automata in the lattice. For recent results on CA dynamics and an up-to-date bibliography see for instance [4–16].

In this paper we study a more general setting relaxing the uniformity constraint. Assume to use CA for simulating a physical or natural phenomenon. Relaxing the uniformity constraint can be justified in several situations:

Generality. In many phenomena, each individual locally interacts with others but maybe these interactions depend on the individual itself or on its position in the space.

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Structural stability. Assume that we are investigating the robustness of a system *w.r.t.* some specific property P. If some individuals change their "standard" behavior does the system still have property P? What is the "largest" number of individuals that can change their default behavior so that the system does not change its overall evolution?

Reliability. CA are more and more used to perform (fast parallel) computations (see for example [2]). Each cell of the CA is implemented by a simple electronic device (FPGAs for example). Then, how reliable are computations w.r.t. failure of some of these devices? (Here failure is interpreted as a device which behaves differently from its "default" way).

Finally, remark that the study of these generalizations has an interest in its own since the new model coincides with the set of continuous functions in Cantor topology.

2 Non-uniform Cellular Automata ν-$\mathcal{CA}$

In the present paper we focus on the one-dimensional case (i.e. the lattice is $\mathbb{Z}$). Remark that most of the following definitions can be easily extended to higher dimensions. Before introducing the formal definition of ν-$\mathcal{CA}$, one should recall the definition of cellular automaton.

Let A be a finite set describing the possible states of any cell. A *configuration* is a snapshot of the states of all cells in the CA, i.e., a function from $\mathbb{Z}$ to A. Given a configuration x, we denote by $x_i \in A$ the state of the cell in position $i \in \mathbb{Z}$. For a fixed state $a \in A$, a configuration is *a-finite* if only a finite number of automata in the CA are in a state different from a.

A (one-dimensional) *CA* is a structure $\langle A, r, f\rangle$, where A is the above introduced finite set of *states*, also called the *alphabet*, and $f : A^{2r+1} \to A$ is the *local rule* whose *radius* is r. The *global rule* induced by a CA $\langle A, r, f\rangle$ is the map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z},\ F(x)_i = f(x_{i-r}, \dots, x_{i+r}) \ . \tag{1}$$

This rule describes the global evolution of the CA from the generic configuration $x \in A^{\mathbb{Z}}$ at time-step $t \in \mathbb{N}$ to the configuration $F(x)$ at the next time-step $t + 1$.

The configuration set $A^{\mathbb{Z}}$ is equipped with the distance $d(x, y) = 2^{-n}$ where $n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$. With respect to the topology induced by d, the configuration set is a Cantor space and F is continuous. Hence, $(A^{\mathbb{Z}}, F)$ is a discrete (time) dynamical system.

Notation. Given a configuration $x \in A^{\mathbb{Z}}$, for any pair $i, j \in \mathbb{Z}$, with $i \leq j$, we denote by $x_{[i,j]}$ the word $x_i \cdots x_j \in A^{j-i+1}$, i.e., the portion of the configuration x inside the interval $[i, j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $[u]_i = \{x \in A^{\mathbb{Z}} : x_{[i,i+k-1]} = u\}$. Cylinders are clopen sets *w.r.t.* the metric d.

The meaning of (1) is that the same local rule f is applied to each site of the CA. Relaxing this last constraint gives us the definition of a ν-$\mathcal{CA}$. More formally one can give the following.

Definition 1 (Non-Uniform Cellular Automaton (ν-$\mathcal{CA}$)). *A* non-uniform Cellular Automaton *(ν-$\mathcal{CA}$) is a structure $\langle A, \{h_i, r_i\}_{i\in\mathbb{Z}}\rangle$ defined by a family of local rules $h_i : A^{2r_i+1} \to A$ of radius r_i all based on the same set of states A.*

Similarly to CA, one can define the global rule of a ν-$\mathcal{CA}$ as the map $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by the law

$$\forall x \in A^{\mathbb{Z}},\ \forall i \in \mathbb{Z}, \quad H(x)_i = h_i(x_{i-r_i}, \ldots, x_{i+r_i}) \ .$$

From now on, we identify a ν-$\mathcal{CA}$ (resp., CA) with the discrete dynamical system induced by itself or even with its global rule H (resp., F).

It is well known that the Hedlund's Theorem [17] characterizes CA as the class of continuous functions commuting with the shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, where $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \sigma(x)_i = x_{i+1}$. It is possible to give a characterization also of the class of ν-$\mathcal{CA}$ since it is straightforward to prove the following.

Proposition 1. *A function $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the global map of a ν-$\mathcal{CA}$ iff it is continuous.*

The previous proposition gives the important information that the pair $(A^{\mathbb{Z}}, H)$ is a discrete dynamical system, but in many practical cases this setting is by far too general to be useful. Therefore, we are going to focus our attention only over two special subclasses of ν-$\mathcal{CA}$.

Definition 2 (dν-$\mathcal{CA}$). *A ν-$\mathcal{CA}$ H is a dν-$\mathcal{CA}$ if there exist two natural k, r and a rule $h : A^{2r+1} \to A$ such that for all integers i with $|i| > k$ it holds that $h_i = h$. In this case, we say that H has default rule h.*

Definition 3 (rν-$\mathcal{CA}$). *A ν-$\mathcal{CA}$ H is a rν-$\mathcal{CA}$ if there exists an integer r such that any local rule h_i has radius r. In this case, we say that H has radius r.*

The first class restricts the number of positions at which non-default rules can appear, while the second class restricts the number of different rules but not the number of occurrences nor it imposes the presence of a default rule. Some easy examples follow.

Example 1. Consider the ν-$\mathcal{CA}$ $H^{(1)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as $\forall x \in A^{\mathbb{Z}}, H^{(1)}(x)_i = 1$ if $i = 0$; 0 otherwise. Remark that $H^{(1)}$ is a dν-$\mathcal{CA}$ which cannot be a CA since it does not commute with σ. So the class of ν-$\mathcal{CA}$ is larger than the original class of all CA.

Example 2. Consider the ν-$\mathcal{CA}$ $H^{(2)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as $\forall x \in A^{\mathbb{Z}}, H^{(2)}(x)_i = 1$ if i is even; 0 otherwise. Remark that $H^{(2)}$ is a rν-$\mathcal{CA}$ but not a dν-$\mathcal{CA}$.

Example 3. Consider the ν-$\mathcal{CA}$ $H^{(3)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as $\forall x \in A^{\mathbb{Z}}, H^{(3)}(x)_i = x_0$. Remark that $H^{(3)}$ is a ν-$\mathcal{CA}$ but not a rν-$\mathcal{CA}$.

Focusing the study on dν-$\mathcal{CA}$ and rν-$\mathcal{CA}$ is not a great loss in generality since (at some extent) each ν-$\mathcal{CA}$ can be viewed as the limit of a sequence of dν-$\mathcal{CA}$.

Proposition 2. *$\mathcal{CA} \subsetneq d\nu\text{-}\mathcal{CA} \subsetneq r\nu\text{-}\mathcal{CA} \subsetneq \nu\text{-}\mathcal{CA}$, where $\mathcal{CA}$ is the set of all CA.*

Proof. The inclusions $\subseteq$ easily follow from the definitions. For the strict inclusions refer to Examples 1 to 3. □

Similarly to what happens for CA one can prove the following.

Proposition 3. *For every rν-$\mathcal{CA}$ H on the alphabet A there exists a radius 1 rν-$\mathcal{CA}$ H' and a bijective continuous mapping ϕ such that $H \circ \phi = \phi \circ H'$. That is, H is topologically conjugated to H'.*

Proof. Let H be a rν-$\mathcal{CA}$. If H has radius $r = 1$ then this result is trivially true using the identity as a conjugacy map. If $r > 1$, let $B = A^r$ and define $\phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ as $\forall i \in \mathbb{Z}, \phi(x)_i = x_{[ir,(i+1)r)}$. Then, it is not difficult to see that the rν-$\mathcal{CA}$ $(B^{\mathbb{Z}}, H')$ of radius 1 defined as $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, H'(x)_i = h'_i(x_{i-1}, x_i, x_{i+1})$ is topologically conjugated to H via ϕ, where $\forall u, v, w \in B, \forall i \in \mathbb{Z}, \forall j \in \{0, \ldots, r-1\}, (h'_i(u, v, w))_j = h_{ir+j}(u_{[j,r)} v w_{[0,j]})$. □

Finally, the following result shows that every rν-$\mathcal{CA}$ is a subsystem of a suitable CA. Therefore, the study of rν-$\mathcal{CA}$ dynamics might reveal new properties for CA and *vice-versa*.

Theorem 1. *Any rν-$\mathcal{CA}$ $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ of radius r is a subsystem of a CA, i.e., there exist a CA $F : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ on a suitable alphabet B and a continuous injection $\phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ such that $\phi \circ H = F \circ \phi$.*

Proof. Consider a rν-$\mathcal{CA}$ $H : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ of radius r. Remark that there are only $n = |A|^{|A|^{2r+1}}$ distinct functions $h_i : A^{2r+1} \to A$. Take a numbering $(f_j)_{1 \le j \le n}$ of these functions and let $B = A \times \{1, \ldots, n\}$. Define the mapping $\phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ such that $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \phi(x)_i = (x_i, k)$, where k is the integer for which $H(x)_i = f_k(x_{i-r}, \ldots, x_{i+r})$. Clearly, ϕ is injective and continuous. Now, define a CA $F : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ using the local rule $f : B^{2r+1} \to B$ such that

$$f((x_{-r}, k_{-r}), \ldots, (x_0, k_0), \ldots, (x_r, k_r)) = (f_{k_0}(x_{-r}, \ldots, x_r), k_0) \ .$$

It is not difficult to see that $\phi \circ H = F \circ \phi$. □

3 CA vs. ν-$\mathcal{CA}$

In this section, we investigate some differences in dynamical behavior between CA and ν-$\mathcal{CA}$. As we are going to see, many characteristics which are really specific to the whole class of CA are lost in the larger class of ν-$\mathcal{CA}$. This will be explored by showing via counter-examples that these properties are not satisfied by the whole class of ν-$\mathcal{CA}$.

First of all, let us recall that given a ν-$\mathcal{CA}$ H, a configuration $x \in A^{\mathbb{Z}}$ is an *ultimately periodic* point of H if there exist $p, q \in \mathbb{N}$ such that $H^{p+q}(x) = H^q(x)$. If $q = 0$, then x is *periodic*, i.e., $H^p(x) = x$. The minimum p for which $H^p(x) = x$ holds is called *period* of x. A ν-$\mathcal{CA}$ is *surjective* (resp. *injective*) if its global rule is surjective (resp. injective).

It is well known that in the case of CA the collection of all ultimately periodic configurations is dense in the configurations space $A^{\mathbb{Z}}$. This property is not true in the general case of ν-$\mathcal{CA}$. We will show this result making reference to the following interesting example of ν-$\mathcal{CA}$.

Example 4. Let $A = \{0, 1\}$ and define the following dν-$\mathcal{CA}$ $H^{(4)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ as

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H^{(4)}(x)_i = \begin{cases} x_i & \text{if } i = 0 \\ x_{i-1} & \text{otherwise} \end{cases} .$$

The first no–go result is relative to the above example.

Proposition 4. *The set of ultimately periodic points of $H^{(4)}$ is not dense.*

Proof. Let $H = H^{(4)}$. Denote by P and U the sets of periodic and ultimately periodic points, respectively. Let $E = \{x \in A^{\mathbb{Z}} : \forall i \in \mathbb{N}, x_i = x_0\}$. Take $x \in P$ with $H^p(x) = x$. Remark that the set $B_x = \{i \in \mathbb{N} : x_i \neq x_0\}$ is empty. Indeed, by contradiction, assume that $B \neq \emptyset$ and let $m = \min B$. It is easy to check that $\forall y \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, H^i(y)_{[0,i]} = {y_0}^{i+1}$, hence $x_m = H^{pm}(x)_m = x_0$, contradiction. Thus $x \in E$ and $P \subseteq E$.

Let $y \in H^{-1}(E)$. We show that $B_y = \emptyset$. By contradiction, let $n = \min B_y$. Since $H(y)_{n+1} = y_n \neq y_0 = H(y)_0$, then $H(y) \notin E$. Contradiction, then $y \in E$ and $H^{-1}(E) \subseteq E$. So $\forall n \in \mathbb{N}, H^{-n}(E) \subseteq E$. Moreover, $U = \bigcup_{n \in \mathbb{N}} H^{-n}(P) \subseteq \bigcup_{n \in \mathbb{N}} H^{-n}(E) \subseteq E$ and E is not dense. □

The following proposition proves that $H^{(4)}$ is not surjective, despite it is based on two local rules each of which generates a surjective CA (namely, the identity CA and the shift CA). Moreover, unlike the CA case (see [17]), $H^{(4)}$ has no configuration with an infinite number of pre-images although it is not surjective.

Proposition 5. *The dν-$\mathcal{CA}$ $H^{(4)}$ is not surjective and any configuration has either 0 or 2 pre-images.*

Proof. Since $\forall x \in A^{\mathbb{Z}}, H^{(4)}(x)_0 = H^{(4)}(x)_1$, configurations in the set $B = \{x \in A^{\mathbb{Z}} : x_0 \neq x_1\}$ have no pre-image. Then any $x \in A^{\mathbb{Z}} \setminus B$ has 2 pre-images y and z such that $\forall i \notin \{-1, 0\}, y_i = z_i = x_{i+1}, y_0 = z_0 = x_0, y_{-1} = 0; z_{-1} = 1$. □

In order to explore some other no-go results, we introduce an other example.

Example 5. Let $A = \{0, 1\}$ and define a ν-$\mathcal{CA}$ $H^{(5)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H^{(5)}(x)_i = \begin{cases} 0 & \text{if } i = 0 \\ x_{i-1} \oplus x_{i+1} & \text{otherwise} \end{cases} ,$$

where $\oplus$ is the xor operator.

The following results show that in the case of ν-$\mathcal{CA}$, the Moore-Myhill theorem on CA [18, 19] is no more true.

Proposition 6. *The ν-$\mathcal{CA}$ $H^{(5)}$ is injective on the finite 0-configurations but it is not surjective.*

Proof. It is evident that $H^{(5)}$ is not surjective. Let x, y be two finite configurations such that $H^{(5)}(x) = H^{(5)}(y)$. By contradiction, assume that $x_i \neq y_i$, for some $i \in \mathbb{Z}$. Without loss of generality, assume that $i \in \mathbb{N}$. Since $x_i \oplus x_{i+2} = H^{(5)}(x)_{i+1} = H^{(5)}(y)_{i+1} = y_i \oplus y_{i+2}$, it holds that $x_{i+2} \neq y_{i+2}$ and, by induction, $\forall j \in \mathbb{N}, x_{i+2j} \neq y_{i+2j}$. We conclude that $\forall j \in \mathbb{N}, x_{i+2j} = 1$ or $y_{i+2j} = 1$ contradicting the assumption that x and y are finite. $\square$

A ν-$\mathcal{CA}$ H is *positively expansive* if there exists $\epsilon > 0$ such for any pair of distinct $x, y \in A^{\mathbb{Z}}$, $d(H^n(x), H^n(y)) > \epsilon$ for some $n \in \mathbb{N}$. A ν-$\mathcal{CA}$ H is *transitive* if for any distinct pair of non-empty open sets $U, V \subset A^{\mathbb{Z}}$, there exists $n \in \mathbb{N}$ such that $H^n(U) \cap V \neq \emptyset$. Both of these properties are considered as standard indicators of chaotic behavior. We will show now that, unlike the CA case, positively expansive ν-$\mathcal{CA}$ are not necessarily transitive nor surjective.

Proposition 7. *The ν-$\mathcal{CA}$ $H^{(5)}$ is positively expansive but it is neither transitive nor surjective.*

Proof. Let $H = H^{(5)}$. By Proposition 6, H is not surjective and hence it is not transitive. Let x and y be two distinct configurations. Without loss of generality, one can assume that there exists $k = \min\{i \in \mathbb{N}, x_i \neq y_i\}$. If $k \leq 1$, we have $d(H^0(x), H^0(y)) \geq \frac{1}{2}$. Otherwise, $H(x)_{k-1} = x_{k-2} \oplus x_k \neq y_{k-2} \oplus y_k = H(y)_{k-1}$ and $H(x)_{[0,k-2]} = H(y)_{[0,k-2]}$. By induction on $k \in \mathbb{N}$, it is easy to see that $H^{k-1}(x)_1 \neq H^{k-1}(y)_1$. Hence $d(H^{k-1}(x), H^{k-1}(y)) \geq \frac{1}{2}$. Thus H is positively expansive with expansivity constant $\frac{1}{2}$. $\square$

Example 6. Let $A = \{0, 1\}$ and define the ν-$\mathcal{CA}$ $H^{(6)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ as follows

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H^{(6)}(x)_i = \begin{cases} x_{i+1} & \text{if } i < 0 \\ x_0 & \text{if } i = 0 \\ x_{i-1} & \text{otherwise.} \end{cases}$$

Recall that for CA, the compactness of $A^{\mathbb{Z}}$ and the uniformity of the local rule allow one to prove that injective CA are surjective. The following result shows that this does not hold in the case of ν-$\mathcal{CA}$.

Proposition 8. *The ν-$\mathcal{CA}$ $H^{(6)}$ is injective but not surjective.*

Proof. Let $H = H^{(6)}$. Concerning non-surjectivity, just remark that only configurations x such that $x_{-1} = x_0 = x_1$ have a pre-image. Let $x, y \in A^{\mathbb{Z}}$ with $H(x) = H(y)$. Then, we have $\forall i > 0, x_{i-1} = y_{i-1}$ and $\forall i < 0, x_{i+1} = y_{i+1}$. So $x = y$ and H is injective. $\square$

4 Surjectivity

In the context of (1D) CA, the notion of De Bruijn graph is very handy to find fast decision algorithms for surjectivity, injectivity and openness. Here, we extend this notion to work with dν-$\mathcal{CA}$ and find decision algorithm for surjectivity. We stress that surjectivity is undecidable for two (or higher) dimensional dν-$\mathcal{CA}$, since surjectivity is undecidable for 2D CA [20].

Definition 4. *Consider a dν-$\mathcal{CA}$ H of radius r and let f be its default rule. Let $k \in \mathbb{N}$ be the largest integer such that $h_k \neq f$ or $h_{-k} \neq f$. The* De Bruijn graph *of H is the pair (V, E) where $V = A^{2r} \times \{-k-1, \ldots, k+1\}$ and E is the set of pairs $((a,i),(b,j))$ with label in $A \times \{0,1\}$ and such that $\forall i \in \{0, \ldots, 2r-1\}, a_i = b_{i+1}$ and one of the following condition is verified*

a) $i = j = -k-1$; in this case the label is $(f(a_0 b), 0)$
b) $i + 1 = j$; in this case the label is $(h_k(a_0 b), 0)$
c) $i = j = k+1$; in this case the label is $(f(a_0 b), 1)$

By this graph, a configuration can be seen as a bi-infinite path on vertexes which passes once from a vertex whose second component is in $[-k+1, k-1]$ and infinite times from other vertices. The second component of vertices allows to single out the positions of local rules different from the default one. The image of a configuration is the sequence of first components of edge labels.

Theorem 2. *Surjectivity is decidable for dν-$\mathcal{CA}$.*

Proof. We show that a dν-$\mathcal{CA}$ H is surjective iff its De Bruijn graph G recognizes the language $(A \times \{0\})^*(A \times \{1\})^*$ when it is considered as the graph of a finite state automaton. Denote by (a_1, a_2) any word of $(A \times \{0,1\})^*$. Let k be as in Definition 4.

Assume that H is surjective and take $u \in (A \times \{0\})^*(A \times \{1\})^*$. Let n be the number of 0's appearing in u (in the second component). We have three cases:

1. If $n = 0$ then there exists $v \in A^*$ such that $f(v) = u$ and we can construct u by the sequence of vertices $(v_{[0,2r)}, k+1), \ldots, (v_{[|v|-2r,|v|)}, k+1)$.
2. If $0 < n < |u|$ then there exists $v \in A^*$ such that $h_{k+1-n}(v) = u$. We can construct u by the sequence of vertices $(v_{[0,2r)}, u_0), \ldots, (v_{[|v|-2r,|v|)}, u_{|v|-2r-1})$ where
$$u_j = \begin{cases} -k-1 & \text{if } k+1-n+j < k \\ k+1 & \text{if } k+1-n+j > k \\ k+1-n+j & \text{otherwise} \end{cases}$$
3. If $n = |u|$ then there exists $v \in A^*$ such that $f(v) = u$ and we can construct u by the sequence of vertices $(v_{[0,2r)}, -k-1), \ldots, (v_{[|v|-2r,|v|)}, -k-1)$.

For the opposite implication, assume that G recognizes $(A \times \{0\})^*(A \times \{1\})^*$. Take $y \in A^{\mathbb{Z}}$ and let $n > k$. Since G recognizes $(y_{[-n,n]}, 0^{n+k+1}1^{n-k})$, there exists $v \in A^*$ such that $H(v)_{[-n,n]} = y_{[-n,n]}$. Set $X_n = \left\{x \in A^{\mathbb{Z}}, x_{[n,n]} = y_{[-n,n]}\right\}$. For any $n \in \mathbb{N}$, X_n is non-empty and compact. Moreover, $X_{n+1} \subseteq X_n$. Therefore, $X = \bigcap_{n \in \mathbb{N}} X_n \neq \emptyset$ and $H(X) = \{y\}$. Hence, H is surjective. □

5 More on Dynamical Properties

5.1 Equicontinuity

Let H be a ν-$\mathcal{CA}$. A configuration $x \in A^{\mathbb{Z}}$ is an *equicontinuity point* for H if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in A^{\mathbb{Z}}$, $d(y, x) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(H^n(y), H^n(x)) < \varepsilon$. A ν-$\mathcal{CA}$ is said to be *equicontinuous* if the set E of all its equicontinuous points is the whole $A^{\mathbb{Z}}$, while it is said to be *almost equicontinuous* if E is residual (i.e., E can be obtained by a countable intersection of dense open subsets). A word $u \in A^k$ is s-blocking ($s \leq k$) for a CA F if there exists an offset $j \in [0, k-s]$ such that for any $x, y \in [u]_0$ and any $n \in \mathbb{N}$, $F^n(x)_{[j,j+s-1]} = F^n(y)_{[j,j+s-1]}$. In [21], Kůrka proved that a CA is almost equicontinuous iff it is non-sensitive iff it admits a blocking word.

We now introduce a class of ν-$\mathcal{CA}$ which will be useful in the sequel. It is an intermediate class between dν-$\mathcal{CA}$ and rν-$\mathcal{CA}$.

Definition 5 (n-compatible rν-$\mathcal{CA}$). *A rν-$\mathcal{CA}$ H is* n-compatible *with a local rule f if for any $k \in \mathbb{N}$, there exist two integers $k_1 > k$ and $k_2 < -k$ such that $\forall i \in [k_1, k_1 + n) \cup [k_2, k_2 + n)$, $h_i = f$.*

In other words, a ν-$\mathcal{CA}$ is n-compatible with f if, arbitrarily far from the center of the lattice, there are intervals of length n in which the local rule f is applied.

The notion of blocking word and the related results cannot be directly restated in the context of ν-$\mathcal{CA}$ because some words are blocking just thanks to the uniformity of CA. To overcome this problem we introduce the following notion.

Definition 6 (Strongly blocking word). *A word $u \in A^*$ is* strongly s-blocking *for a CA F of local rule f if there exists an offset $p \in [0, |u| - s]$ such that for any ν-$\mathcal{CA}$ H with $\forall i \in \{0, \ldots, |u| - 1\}$, $h_i = f$ it holds that*

$$\forall x, y \in [u]_0, \forall n \geq 0, \quad H^n(x)_{[p,p+s)} = H^n(y)_{[p,p+s)} \quad .$$

Roughly speaking, a word is strongly blocking if it is blocking whatever be the perturbations involving the rules in its neighborhood. The following extends Proposition 5.12 in [22] to strongly r-blocking words.

Proposition 9. *Any r radius CA F is equicontinuous iff there exists $k > 0$ such that any word $u \in A^k$ is strongly r-blocking for F.*

Proof. If any word is strongly blocking then F is obviously equicontinuous. For the opposite implication, by [22, Prop. 5.12], there exist $p > 0$ and $q \in \mathbb{N}$ such that $F^{q+p} = F^q$. As a consequence, we have that $\forall u \in A^*, |u| > 2(q+p)r \Rightarrow f^{p+q}(u) = f^q(u)_{[pr,|u|-(2q+p)r)}$. Let H be a ν-$\mathcal{CA}$ such that $h_i = f$ for each $i \in \{0, \ldots, (2p+2q+1)r-1\}$. For any $x \in A^{\mathbb{Z}}$ and $i \geq 0$, consider the following words: $s^{(i)} = H^i(x)_{[0,qr)}$, $t^{(i)} = H^i(x)_{[qr,(q+p)r)}$, $u^{(i)} = H^i(x)_{[(q+p)r,(q+p+1)r)}$, $v^{(i)} = H^i(x)_{[(q+p+1)r,(q+2p+1)r)}$, $w^{(i)} = H^i(x)_{[(q+2p+1)r,(2q+2p+1)r)}$. For all $i \in 0, \ldots, q+p$, $u^{(i)}$ is fully determined by $s^{(0)}t^{(0)}u^{(0)}v^{(0)}w^{(0)} = x_{[0,(2q+2p+1)r)}$. Moreover, for any natural i, we have $u^{(i+q+p)} = f^{q+p}(s^{(i)}t^{(i)}u^{(i)}v^{(i)}w^{(i)}) =$

$f^q(s^{(i)}t^{(i)}u^{(i)}v^{(i)}w^{(i)})_{[pr,(p+1)r)} = (t^{(i+q)}u^{(i+q)}v^{(i+q)})_{[pr,(p+1)r)} = u^{(i+q)}$. Summarizing, for all $i \in \mathbb{N}$, $u^{(i)}$ is determined by the word $x_{[0,(2q+2p+1)r)}$ which is then strongly r-blocking. Since x had been chosen arbitrarily, we have the thesis. $\square$

Theorem 3. *Let F be a CA with local rule f admitting a strongly r-blocking word u. Let H be a $r\nu$-$\mathcal{CA}$ of radius r. If H is $|u|$-compatible with f then H is almost equicontinuous.*

Proof. Let p and n be the offset and the length of u, respectively. For any $k \in \mathbb{N}$, consider the set $T_{u,k}$ of configurations $x \in A^{\mathbb{Z}}$ having the following property $\mathcal{P}$: there exist $l > k$ and $m < -k$ such that $x_{[l,l+n)} = x_{[m,m+n)} = u$ and $\forall i \in [l, l+n) \cup [m, m+n)$ $h_i = f$. Remark that $T_{u,k}$ is open, being a union of cylinders. Clearly, each $T_{u,k}$ is dense, thus the set $T_u = \bigcap_{k \geq n} T_{u,k}$ is residual. We claim that any configuration in T_u is an equicontinuity point. Indeed, choose arbitrarily $x \in T_u$. Set $\epsilon = 2^{-k}$, where $k \in \mathbb{N}$ is such that $x \in T_{u,k}$. Then, there exist $k_1 > k$ and $k_2 < -k - n$ satisfying $\mathcal{P}$. Fix $\delta = \min\{2^{-(k_1+n)}, 2^{-k_2}\}$ and let $y \in A^{\mathbb{Z}}$ be such that $d(x, y) < \delta$. Then $y_{[k_2,k_1+|u|)} = x_{[k_2,k_1+|u|)}$. Since u is r-blocking, $\forall t \in \mathbb{N}$, $H^t(x)$ and $H^t(y)$ are equal inside the intervals $[k_1+p, k_1+p+r]$ and $[k_2+p, k_2+p+r]$, then $d(H^t(x), H^t(y)) < \epsilon$. $\square$

In a similar manner one can prove the following.

Theorem 4. *Let F be an equicontinuous CA of local rule f. Let $k \in \mathbb{N}$ be as in Proposition 9. Any $r\nu$-$\mathcal{CA}$ k-compatible with f is equicontinuous.*

5.2 Sensitivity to the Initial Conditions

Recall that a CA F is *sensitive to the initial conditions* (or simply *sensitive*) if there exists a constant $\varepsilon > 0$ such that for any configuration $x \in A^{\mathbb{Z}}$ and any $\delta > 0$ there is a configuration $y \in A^{\mathbb{Z}}$ such that $d(y, x) < \delta$ and $d(F^n(y), F^n(x)) \geq \varepsilon$ for some $n \in \mathbb{N}$.

Example 7. Let $A = \{0, 1, 2\}$ and consider the CA whose local rule $f : A^3 \to A$ is defined as follows: $\forall x, y \in A$, $f(x, 0, y) = 1$ if $x = 1$ or $y = 1$, 0 otherwise; $f(x, 1, y) = 2$ if $x = 2$ or $y = 2$, 0 otherwise; $f(x, 2, y) = 0$ if $x = 1$ or $y = 1$, 2 otherwise.

Proposition 10. *The CA defined in Example 7 is almost equicontinuous.*

Proof. Just remark that the number of 0s inside the word 20^i2 is non-decreasing. Thus 202 is a 1-blocking word. $\square$

The following example defines a ν-$\mathcal{CA}$ which is sensitive to the initial conditions although its default rule give rise to an almost equicontinuous CA.

Example 8. Consider the dν-$\mathcal{CA}$ $H^{(8)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined as follows

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H^{(8)}(x)_i = \begin{cases} 1 & \text{if } i = 0 \\ f(x_{i-1}, x_i, x_{i+1}) & \text{otherwise} \end{cases},$$

where f and A are as in Example 7.

Remark that positive and negative cells do not interact each other under the action of $H^{(8)}$. Therefore, in order to study the behavior of $H^{(8)}$, it is sufficient to consider the action of $H^{(8)}$ on $A^{\mathbb{N}}$.

Lemma 1. *For any $u \in A^*$, $\exists n_0 \in \mathbb{N}$ such that $\forall n > n_0, (H^{(8)})^n(u0^\infty)_1 = 1$.*

Lemma 2. $\forall u \in A^*$*,* $\forall n_0 \geq 0, \exists n > n_0, (H^{(8)})^n(u2^\infty)_1 = 2$.

Proposition 11. *The dν-$\mathcal{CA}$ $H^{(8)}$ is sensitive.*

Proof. Let $H = H^{(8)}$ and $\mathcal{F}$ be the set of all a-finite configurations for $a \in \{0, 2\}$. By a theorem of Knudesen [23], we can prove the statement *w.r.t.* $\mathcal{F}$. Then, for any $u \in A^*$. Build $x = u0^\infty$ and $y = u2^\infty$. By Lemma 1 and 2, there exists n such that $1 = H^n(x)_1 \neq H^n(y)_1 = 2$. And hence H is sensitive with sensitivity constant $\epsilon = 1/2$. □

The following example shows that default rules individually defining almost equicontinuous CA can also constitute ν-$\mathcal{CA}$ that have a completely different behavior from the one in Example 8.

Example 9. Let $A = \{0, 1, 2\}$ and define the local rule $f : A^3 \to A$ as: $\forall x, y, z \in A$, $f(x, y, z) = 2$ if $x = 2$ or $y = 2$ or $z = 2$, z otherwise. The CA F of local rule f is almost equicontinuous since 2 is a blocking word. The restriction of F to $\{0, 1\}^{\mathbb{Z}}$ gives the shift map which is sensitive. Thus F is not equicontinuous. Define now the following dν-$\mathcal{CA}$ $H^{(9)}$:

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad H^{(9)}(x)_i = \begin{cases} 2 & \text{if } i = 0 \\ f(x_{i-1}, x_i, x_{i+1}) & \text{otherwise} \end{cases}.$$

Proposition 12. *The dν-$\mathcal{CA}$ $H^{(9)}$ is equicontinuous.*

Proof. Let $n \in \mathbb{N}, x, y \in A^{\mathbb{Z}}$ be such that $x_{[-2n,2n]} = y_{[-2n,2n]}$. Since H is of radius 1, $\forall k \leq n, H^k(x)_{[-n,n]} = H^k(y)_{[-n,n]}$ and $\forall k > n$, $H^k(x)_{[-n,n]} = 2^{2n+1} = H^k(y)_{[-n,n]}$. So, H is equicontinuous. □

5.3 Expansivity and Permutivity

Recall that a rule $f : A^{2r+1}$ is *leftmost* (resp., rightmost) *permutive* if $\forall u \in A^{2r}, \forall b \in A, \exists! a \in A, f(au) = b$ (resp., $f(ua) = b$). This definition can be easily extended to ν-$\mathcal{CA}$. Indeed, we say that a rν-$\mathcal{CA}$ is leftmost (resp. rightmost) permutive if all h_i are leftmost (resp. rightmost) permutive. A ν-$\mathcal{CA}$ is *permutive* if it is leftmost or rightmost permutive.

In a very similar way to CA, given a ν-$\mathcal{CA}$ H and two integers $a, b \in \mathbb{N}$ with $a < b$, the *column subshift* $(\Sigma_{[a,b]}, \sigma)$ of H is defined as follows $\Sigma_{[a,b]} = \{y \in \left(A^{b-a+1}\right)^{\mathbb{N}} : \exists x \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, y_i = H^i(x)_{[a,b]}\}$. Consider the map $\mathcal{I}_{[a,b]} : A^{\mathbb{Z}} \to \Sigma_{[a,b]}$ defined as $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, \mathcal{I}_{[a,b]}(x)_i = H^i(x)_{[a,b]}$. It is not difficult to see that $\mathcal{I}$ is continuous and surjective. Moreover $H \circ \mathcal{I}_{[a,b]} = \mathcal{I}_{[a,b]} \circ \sigma$. Thus (Σ_I, σ) is a *factor* of the ν-$\mathcal{CA}$ $(A^{\mathbb{Z}}, H)$ and we can lift some properties from $(\Sigma_{[a,b]}, \sigma)$ to $(A^{\mathbb{Z}}, H)$. The following result tells that something stronger happens in the special case of leftmost and rightmost permutive rν-$\mathcal{CA}$.

Theorem 5. *Any leftmost and rightmost permutive rν-$\mathcal{CA}$ of radius r is conjugated to the full shift $((A^{2r})^{\mathbb{N}}, \sigma)$.*

Proof. Just remark that the map $\mathcal{I}_{[1,2r]} : A^{\mathbb{Z}} \to (A^{2r})^{\mathbb{N}}$ is bijective. □

The requirements of the previous theorem are very strong. Indeed, there exist ν-$\mathcal{CA}$ which are topologically conjugated to a full shift but that are not permutive. As an example, consider the ν-$\mathcal{CA}$ H defined as follows

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, H(x)_i = \begin{cases} x_{i-1} \text{ if } i \leq 0 \\ x_{i+1} \text{ otherwise} \end{cases} .$$

Then, $\Sigma_{[0,1]} = (A^2)^{\mathbb{N}}$ et $\mathcal{I}_{[0,1]}$ is injective.

6 Conclusions

In this paper we started exploring the dynamical behavior of ν-$\mathcal{CA}$. Many specific properties for CA are no longer true for ν-$\mathcal{CA}$. However, under certain conditions, some stability forms turned out to be quite robust when altering a CA to obtain a ν-$\mathcal{CA}$. Despite of the many no-go results proved in this paper, we strongly believe that ν-$\mathcal{CA}$ can be useful for many practical applications and hence deserve further studies.

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